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ARTICLE

Promote the Compression Efficiency of Digital Images by Using Improved CUR Matrix Decomposition Algorithm

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ABSTRACT

In order to overcome the problem that the CUR matrix decomposition algorithm loses a large amount of information when compressing images, the quality of reconstructed images is not high, we propose a CUR matrix decomposition algorithm based on standard deviation sampling. Because of retaining more image information, the reconstructed image quality is higher under the same compression ratio. At the same time, in order to further reduce the amount of image information lost during the sampling process of the CUR matrix decomposition algorithm, we propose the SVD-CUR algorithm. The experimental results verify that our algorithm can achieve high image compression efficiency, and also demonstrate the high precision and robustness of CUR matrix decomposition algorithm in dealing with low rank sparse matrix data.

1. Introduction

With the advent of the digital age, large amounts of image information can put tremendous pressure on the storage capacity of the memory, the bandwidth of the communication trunk channel, and the processing speed of the calculation. To solve these problems, we need to compress the image data during its transmission and storage. Commonly used image compression coding methods can be divided into lossy coding and lossless coding. Since lossy compression generally achieves a higher compression ratio, it is an ideal coding option when the quality of the image is not very high.

We often represent the grayscale image as a two-dimensional array (matrix), and RGB images are represented by an M×N×3 multidimensional data matrix. Thus, our processing of digital images can be, in a sense, an operation on the digital matrix of images. In recent years, with the continuous development and improvement of applied mathematics, the application of matrix algebra in various subject areas has become more and more extensive. In 2006, a digital image compression coding method based on matrix singular value decomposition (SVD)[1,2] theory appeared.[3] The basic idea is to transform the original image matrix into a low-rank approximation matrix by the singular value decomposition technique of the matrix, reduce the amount of data needed to represent the image, and reduce the resources occupied by the image. In addition, SVD plays an important role in large-scale matrix dimensionality reduction, such as in complex networks,[4]

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feature discovery.[5-7]

The CUR decomposition of the matrix[8-10] can avoid the low-rank approximation of the original matrix by avoiding the eigenvalue solution for the high-dimensional matrix.[11-14] We consider applying it to image data compression. For the characteristics of image data, we propose a CUR matrix decomposition algorithm based on standard deviation sampling. Furthermore, we combine CUR and SVD, two matrix decomposition-based data reduction algorithms, which effectively overcome the shortcomings of the CUR decomposition algorithm and can greatly reduce the scale of the image.

2. Matrix CUR Decomposition Algorithm and Its Improvement

2.1 Matrix CUR Decomposition

2.1.1 Definition 1

Let A be a matrix of m×n, and C be an m×c (general c < n) submatrix of A, R is an r×n (general r < m) submatrix of A, U is a c×r matrix, then matrix $\widetilde{A}$ is a low rank approximation (CUR approximation) based on row and column selection of A, i.e.

$$A \approx \widetilde{A} = CUR \quad (1)$$

Where U is a normal matrix, U=C+AR+, and C+ and R+ are respectively the Moore-Penrose inverses of C and R. The CUR decomposition structure of the matrix can be simply expressed as follows:

Figure 1. The schematic diagram of matrix CUR decomposition

2.1.2 Definition 2

If there are linear equations (in Moore-Penrose inverses): $Ax=b$, $A \in C^{m\times n}$, $b \in Cm$, and $X \in C^{m\times n}$, and if there is an arbitrary b, the linear equations have solutions: $x=Xb$, $X=A^{+}b$, then matrix $A^{+} \in C^{m\times n}$ is called the Moore-Penrose inverse of A. Obviously, the above definition must also be true for the real matrix.

2.2 Algorithm 1: CUR Matrix Decomposition Algorithm

Input: the original image matrix $A \in R^{m\times n}$

Output: the approximate matrix $\widetilde{A} \in R^{m\times n}$

(1) Randomly extract c ($c \leq n$) columns from A to form submatrix $C \in Rm\times c$;

(2) Randomly extract r ($r \leq m$) rows from A to form submatrix $R \in Rr\times n$;

(3) Calculate the normal matrix $U=C^{+}AR^{+}$, in which C^{+} and R^{+} are respectively the Moore-Penrose inverses of C and R;

(4) Return to the approximate matrix $\widetilde{A}=CUR$ of the original matrix A.

2.3 Improved CUR Matrix Decomposition Algorithm

The CUR matrix decomposition algorithm has strong randomness and instability in the process of constructing submatrices C and R, which makes the constructed matrices U and $\widetilde{A}$ uncertain, the error is large, and the randomly extracted columns or rows, due to their atypicality, will cause the loss of many important information of the original image matrix.

We select an RGB image of jpg[15-16] format of 512×512×3 from the international standard test image set and convert it into a double-precision gray image, as shown in Figure 2. Then we use this image as the original image of the subsequent simulation experiment (in the MATLAB environment).

Figure 2. The 512 × 512 original Lena image

We calculate the characteristics of the original image matrix row and column standard deviation distribution as follows.

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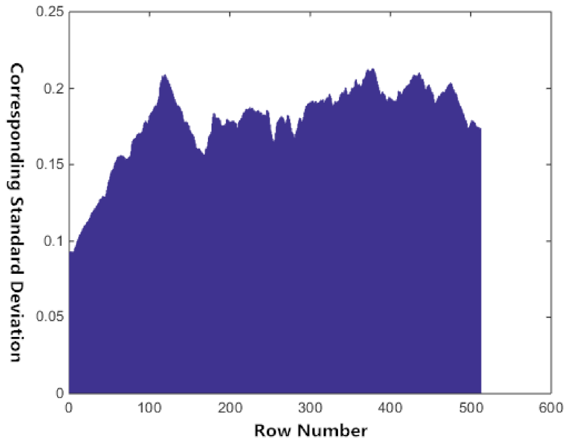
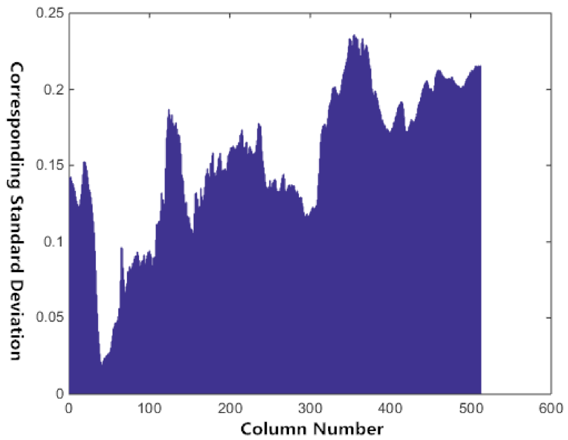


(a) The row standard deviation



(b) The column standard deviation

Figure 3. Image matrix row and column standard deviation distribution bar graph

As shown in Figure 3, the distribution of the standard deviation of the row and column of the image matrix is very uneven, and the information of the image contained in the row or column with a large standard deviation is large, and the contribution to the image feature is also large. In the CUR decomposition, we construct the sub-matrices R and C by extracting the rows and columns with the largest standard deviation to preserve as much original image information as possible, so that the compressed image quality is higher. The specific algorithm steps are as follows:

Algorithm 2: CUR Decomposition Algorithm Based on Standard Deviation Sampling

Input: the original image matrix $A \in R^{m\times n}$

Output: the approximate matrix $\widetilde{A} \in R^{m\times n}$

(1) Calculate the row and column standard deviation of the original matrix A;

(2) Sort the row and column standard deviations in step (1) in order from largest to smallest;

(3) Select the columns and rows corresponding to the first c and the first r maximum standard deviations respectively, and construct the submatrices C and R;

(4) Construct U according to step (3) of Algorithm 1;

(5) Return to the approximate matrix A according to step (4) of Algorithm 1.

3. CUR Algorithm Combining SVD

3.1 The Basic Principle of Matrix Singular Value Decomposition (SVD)

Principle 1: Let $A \in Rm\times n$, there are orthogonal matrices $U=[u1, u2, ...,um] \in Rm\times m$, $V=[v1, v2, ...,vn] \in Rn\times n$, and $\{u^t\}_{t=1}^{m} \in R^m, \{v^t\}_{t=1}^{n} \in R^n$, which make that,

$$A = U\Sigma V^T \tag{2}$$

Where

$\Sigma = diag(\sigma_1,\cdots,\sigma_\rho) \in R^{m\times n}, \rho = \min\{m,n\}, \sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_\rho \ge 0, \sigma_i$ is called the singular value of A, and vectors ui and vi are called the i-th left and right singular vectors, respectively.

If $k \le r = rank(A)$, we define $A_k = U_k \Sigma_k V_k^T = \Sigma_{t=1}^{k} \sigma_t u^t v^{tT}$.

The singular values of any image matrix satisfy the "large L curve" as shown in Figure 4. Larger singular values only account for a small fraction of all singular values. We choose these larger k ($k < r$) singular values to approximate the original matrix A, i.e. $Ak \approx A$, which achieves the effect of reducing the rank.

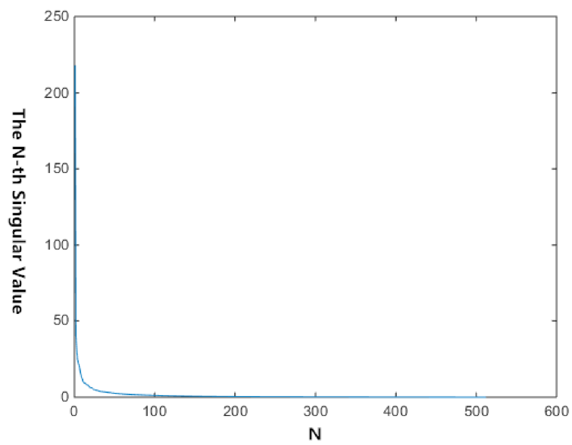


Figure 4. Singular value characteristic curve of the original image

3.2 SVD-CUR Algorithm

Since the image matrix is generally a full rank matrix, direct sampling will lead to a large lack of image informa-

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tion. We first perform SVD decomposition on the original image matrix, and the reduced rank, reduced rank approximation retains most of the original image. Then, the CUR decomposition of the standard deviation sampling is performed on the low rank matrix, which further reduces the image size, so that the information loss of the reconstructed image is less, and the quality of the reconstructed image is improved. The specific algorithm steps are as follows:

Algorithm 3: SVD-CUR Algorithm

Input: the original image matrix $A \in Rm\times n$

Output: the approximate matrix $\widetilde{A} \in Rm\times n$

(1) Perform SVD decomposition on A;

(2) Select the first k larger singular values and the corresponding singular vectors to construct a low rank matrix Ak;

(3) Gradually further process Ak according to Algorithm 2;

(4) Return to the approximate matrix $\widetilde{A} \in Rm\times n$.

4. Accuracy and Compression Ratio Analysis Based on CUR Decomposition Method

4.1 Accuracy

The error produced by the original matrix after CUR decomposition can be measured by the root mean square distortion ratio (PRD):

$$PRD(\%) = \frac{\|A - CUR\|_F}{\|A\|_F} \tag{3}$$

Where $\|A\|_F = \left(\sum_i \sum_j |a_{ij}|^2 \right)^{\frac{1}{2}}$ is called the Frobenius norm of the matrix.

The root-mean-square distortion ratio is a widely used test method for comparing the reconstruction performance of low-dimensional models. In some specific cases, the error rate is almost zero.

4.2 Compression Ratio

We calculate the ratio of the total number of elements in the original (uncompressed) matrix to the total number of elements in the corresponding low rank approximation matrix by calculating the Compression Ratio (CR).

4.2.1 Compression Ratio of CUR Decomposition

In the matrix CUR decomposition, we approximate the original matrix A by $C \in Rm\times r$, $U \in RC\times r$ and $R \in Rr\times n$, at which point the compression ratio is:

$$Cr_1 = \frac{mn}{mc + cr + rn} \tag{4}$$

Where c and r respectively represent the number of columns and rows stored in the low rank approximation image matrix. The compression ratio formula for CUR decomposition can be improved by the following method:

In the CUR decomposition, we store the original columns and subsets of the data matrix A with C and R, respectively. Obviously, C and R have cr elements that are identical. To improve compression efficiency, we replace the cr redundant data by storing $c+r$ positions of these columns and rows. Therefore, the compression ratio based on CUR decomposition can be optimized to:

$$Cr_2 = \frac{mn}{mc + cr + (rn - cr) + (c + r)} \tag{5}$$

Later, we will use Formula (5) to calculate the compression ratio of the CUR decomposition. It is easy to obtain from this formula, and the larger the $c+r$ is, the larger the number of rows and columns extracted from the original image matrix, the smaller the compression ratio, and the smaller the amount of data compressed by the image. Conversely, the fewer the number of rows and columns extracted, the larger the compression ratio, and the more data the image compresses.

4.2.2 Compression Ratio of SVD-CUR Algorithm

At first, we define the compression ratio of the preprocessed image after SVD decomposition:

$$Cr_3 = \frac{mn}{(m + n + 1)k} \tag{6}$$

Where m and n respectively represent the row and column number of the image data before compression and k represents the number of larger singular values selected.

The compression ratio Cr of the SVD-CUR algorithm is defined as the sum of the compression ratio of the matrix SVD decomposition and the compression ratio of the CUR decomposition, i.e.

$$Cr = Cr_2 + Cr_3 \tag{7}$$

5. Experimental Results and Analysis

5.1 Improved CUR Algorithm Experiment

(a)

(b)

(c)

(d)

(e)

(f)

Figure 5. CUR and its improved algorithm image compression effects

As shown in Figure 5, (a), (b), and (c) are images reconstructed from the digital matrix of Figure 2 by extracting 250, 200, 150 rows and columns according to the sampling method in Algorithm 2; Figures (d), (e), and (f) are images

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reconstructed after randomly extracting the same number of rows and columns according to Algorithm 1. Obviously, the quality of the reconstructed image of Algorithm 2 is better than Algorithm 1, and the sharpness of the image is gradually decreasing as the number of samples is reduced.

5.2 Experiment of SVD-CUR Algorithm

The simulation experiment of 5.1 shows that the improved CUR algorithm can achieve certain image compression effects when the number of rows and columns is large, but it is not ideal. In the experiment of SVD-CUR algorithm below, after SVD decomposition preprocessing is performed on the image matrix of Figure 2, we carry out the experiment of CUR algorithm by extracting the smaller number of rows and columns, in order to further improve the compression ratio of the image. At the same time, in order to test the sensitivity of the CUR algorithm to the matrix rank before sampling, we divided the experiments into two groups. The ranks of the image matrices before the CUR algorithm were different in the two groups. The simulation results are as follows:

(a)

(b)

(c)

(d)

(e)

Group 1 (when k=64)

(f)

(g)

(h)

(i)

(j)

Group 2 (when k=44)

Figure 6. SVD-CUR algorithm image compression effects

Table 1. Partial statistical characteristics of reconstructed images (Group 1)

Statistical indicators	SVD Decomposition Initialization Image	Sampling Number of Rows and Columns			
		100	80	60	40
Entropy	7.4274	7.4274	7.4274	7.4802	7.4373
Gray average	0.4154	0.4154	0.4154	0.4146	0.4017
Gray Standard Deviation	0.1839	0.1823	0.1823	0.1784	0.1689
Compression Ratio	3.9961	6.1519	6.6908	7.5891	9.3856

Table 2. Partial statistical characteristics of reconstructed images (Group 2)

Statistical indicators	SVD Decomposition Initialization Image	Sampling Number of Rows and Columns			
		100	80	60	40
Entropy	7.4396	7.4396	7.4396	7.4396	7.4857
Gray Average	0.4154	0.4154	0.4154	0.4154	0.4123

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Gray Standard Deviation	0.1839	0.1811	0.1811	0.1811	0.1768
Compression Ratio	5.8125	7.9683	8.5072	9.4055	11.2020

As shown in FIG. 6, the first group of images is (a)-(e), and the figure (a) is an image reconstructed according to the SVD decomposition principle of the matrix, taking =64 (i.e., singular value less than 2 is ignored). Figures (b)-(e) are images reconstructed sequentially by extracting 100, 80, 60, 40 rows and columns from the original image matrix (a) according to Algorithm 3; the second group of images is (f)-(i), Figure (f) is an image reconstructed with =44 (i.e., singular values less than 3 are ignored). Figures (g)-(j) are images reconstructed according to Algorithm 3 using the same number of samples in Group 1 respectively.

The quality of (a) in the first group is slightly higher than that in the second group (f), which is because when we perform SVD decomposition on the digital matrix of Figure 2, the number of singular values is more selected when reconstructing the image, and more information of the original image is retained.

Although the number of samples in these two groups is relatively small, the quality of image restoration is ideal. The first three images (a), (b), and (c) of the first group are almost indistinguishable, and the first four images (f), (g), (h), and (i) of the second group are almost indistinguishable. Compression is almost "lossless". In contrast, the image restoration quality of the first group decreases more rapidly with the decrease of the number of sample rows and columns, and the image quality of the second group decreases relatively well with the decrease of the number of sample rows and columns which shows that the CUR algorithm has better robustness to low rank matrices.

As shown in Tables 1 and 2, the image quality evaluation results obtained by the naked eye observation image are basically consistent with the partial objective metrics of the reconstructed image, and as the rank of the sampled image matrix decreases and the number of samples decreases, and then we can get a better compression ratio.

6. Conclusion

In order to preserve the image information reconstructed by the traditional CUR matrix decomposition algorithm, we propose a CUR matrix decomposition algorithm based on standard deviation sampling. The most basic evaluation criteria for analyzing the accuracy and compression ratio of CUR matrix decomposition algorithm are also given. Furthermore, we combine the matrix singular value decomposition algorithm and the CUR matrix decomposition algorithm to preprocess the original image matrix by singular value decomposition, and then use the improved CUR matrix decomposition algorithm. The experimental results show that the reconstructed image quality of CUR matrix decomposition algorithm based on standard deviation sampling is higher than the traditional CUR matrix decomposition algorithm, and the approximate image reconstructed by SVD-CUR algorithm can obtain a larger compression ratio, and the image compression effect is ideal and it is still very stable, and it also shows that the CUR algorithm is more suitable for the decomposition processing of low rank sparse matrices.

Next, we consider compressing the face image using the algorithm proposed in this paper, and using the compressed image as the pre-processed image, then performing image segmentation and feature extraction on them, and using pattern recognition technology to classify the face image of the batch. On the other hand, we will also explore the use of this algorithm for image data in specific areas, such as remote sensing images and medical images.

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