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Three Methods to Calculate the Financial Risk Measurement: Value-At-Risk and Expected Shortfall

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ABSTRACT

This paper analyzes the relationship between the risk factor of each stock and the portfolio's risk based on a small portfolio with four U.S. stocks, and the reason why these risk factors can be regarded as a market invariant. Then, it evaluates the properties of the convex and coherent risk indicators of the capital requirement index composed of VaR and ES, and use three methods(the historical estimation method, boudoukh's mixed method and Monte Carlo method) to estimate the risk measurement indicators VaR and ES respectively based on the assumption of multivariate normal distribution' risk factors and multivariate student t-copula distribution's one, finally it figures out that these three calculation results are very close.

1. Exploratory Analysis of Risk Factors

Section 1 gives a basic description of statistical properties of risk factors Xi for all four assets and an explanation of how log-returns reflect the riskiness of both individual asset and the whole portfolio. And it presents reasons why risk factors are market invariants based on the method originated by Attilio Meucci[1].

From table 1, we can verify clearly that the average log-returns of stock 1 and stock 2 are negative, whereas stock 3 and 4 have positive average risk factors, which means the return of asset 1 and asset 2 have lower averaging benefits[4].

Table 1. Mean, Variance, Min, Max, Quartiles of 4 Risk Factors

Risk factors	X1	X2	X3	X4
Mean	-8.493145e-05	-8.935721e-05	8.845965e-05	8.474993e-05
Variance	0.0003200096	0.0002307729	0.0002749404	0.0007659206
Minimum	-0.1155822	-0.150271	-0.180606	-0.2876821
Maximum	0.1572141	0.1586307	0.1325256	0.2586502
1st quartile	-0.0082076046	-0.0068018315	-0.007854612	-0.01035043
2nd quartile	0.0005981756	-0.0001143053	0.000000000	0.00000000
3rd quartile	0.0084305330	0.0068594796	0.008084315	0.01040593

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For variances, the smallest is for stock 2 and the largest is for stock 4. They indicate that the second stock is the most stable one during this period but the fourth oscillates more frequently and represents high riskiness. Stock 4 is the riskiest one among these four assets. The more proportion of stock 4 results in riskier portfolio.

The maximal log return (0.2586502) is for stock 4. It is apparently higher than those of other three assets. From the combination of the maximum and minimum of stock 4, we can realize that it is the riskiest one as well. Since stock 4 occupies 46.67% of the whole portfolio, the fluctuations of asset 4 connect to the riskiness of portfolio closely.

Table 2. Covariances between risk factors

Covariance	X1	X2	X3	X4
X1	0.0003200096	0.0002054446	0.0001573925	0.0001872091
X2	0.0002054446	0.0002307729	0.0001374079	0.0001568926
X3	0.0001573925	0.0001374079	0.0002749404	0.0001914346
X4	0.0001872091	0.0001568926	0.0001914346	0.0007659206

From table 2, we can see that all covariances are positive. It means that when one log return increases, other three will increase as well. All four stocks show the same trend of riskiness changes. And X1 is more correlated to X2. The connection between X3 and X4 is closer. For the whole portfolio, the riskiness changes with same direction of risk factor changes.

Modelling the stock market requires there to be a repetitive statistical behavior, such that the stock prices are independent and identically distributed (i.i.d) over time, this is called in market invariants. If the log-returns of the stocks, Xi for i = 1,2,3,4, are market invariant then it implies that any shocks to the market will be short-term and the readjust to market behavior over time. Also, it allows predictions to be made about future outcomes with a certain confidence level. According to Meucci[1] (2005) there are two simple graphical tests we can use detect invariance in our time series. In the first test, we divide the time series Xi into two separate series, then we plot histograms of the two series. If Xi is market invariant then the histogram of the 1st half-sample should resemble the histogram of the 2nd half-sample as shown in figure 1.

The second test for invariance does not require splitting the data but rather plotting the entire time series against its lagged values. If is market invariant then the scatter plots for i = 1,2,3,4 should resemble a circular cloud (Meucci[1], 2005). In figure 2 we see that this is the case.

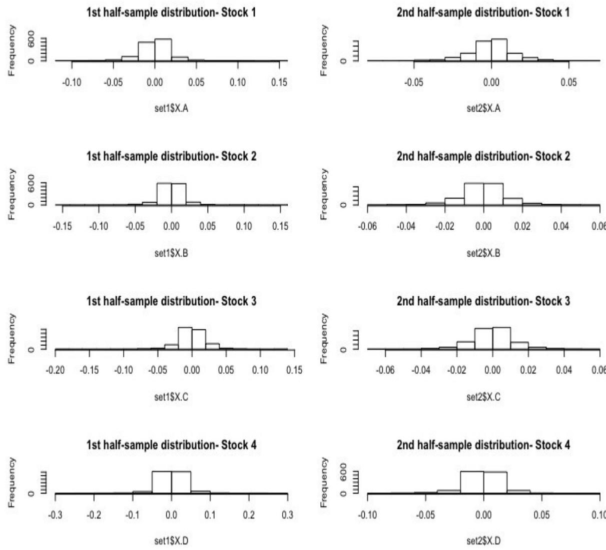

Figure 1. Stock Logarithmic Returns

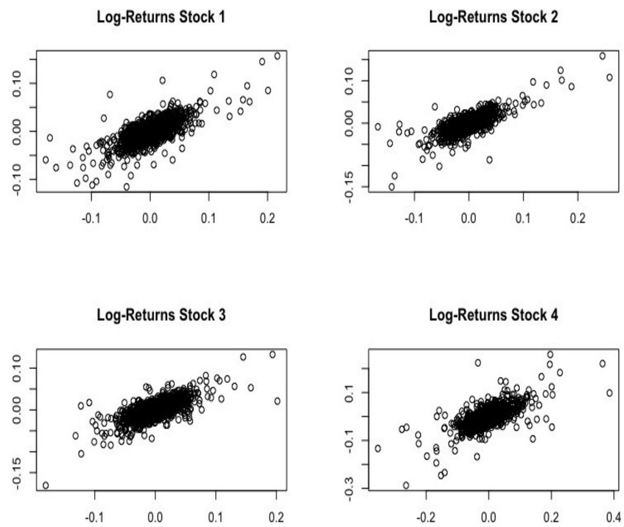

Figure 2. Scatter plot with lags of log-returns for Stocks: 1, 2, 3, 4

Figure 2: Testing for Normality: In theory, total returns are assumed to be log-normally distributed, therefore, if that is true then we can also assume that the log-returns, Xi follows a normal distribution (Meucci, 2005). We can check if Xi is normally distributed by using a normal Q-Q plot which plots the standardized empirical quantiles of the observed data against the quantile of a standard normal random variable. Normality on a Q-Q plot is represented by a dense scatter of points at 45◦ around the line of best fit. In figure 3 we can see that our Q-Q plots show a tight variability around the middle of the data but there are great deviation on the tail ends. Under regular circumstances these Q-Q plots would allow us to reject the assumption of

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normality but in this case the greater variability at the tail ends of the points is caused by volatility clustering, which means that volatility is higher in some periods such as the 2008 financial crisis. To confirm whether or not our log-returns Xi follow normally distributed, we can use the Jarque-Bera normality test which determines if Xi have the skewness and kurtosis that matches a normal distribution. The test yielded a p-value < 2.2 × 10−16 for each log-returns, therefore we cannot assume Xi to be normally distributed.

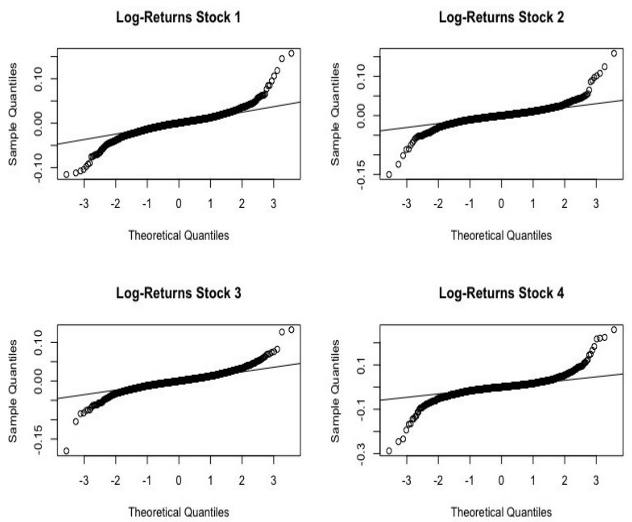

Figure 3. Normal Q-Q plots of log-returns for Stocks: 1, 2, 3,4

2. Estimating Value at Risk & Expected Shortfall Using Historical-Estimation Method

In this section, we apply the historical-estimation method and hybrid method in the paper by Boudoukh et al (1998) in section 2 to calculate the value at risk (VaR) and expected shortfall (ES) at the confidence level 99.9%.[2]

Value at Risk (VaR) and Expected Shortfall (ES) are commonly used risk measures in analyzing market risk. VaR and ES both measures the potential loss of our portfolio over a period of time for a given confidence interval. VaR is the worse-case loss of our portfolio over a period of time, whereas ES is the average loss that occur after the worse-case VaR threshold. It could also be used in calculating the minimal capital requirement that our investment fund is required to hold to have protection against the risks we are taking on.

In this part we will take a look at computing VaR and ES using the historical-estimation method. This method uses historical data for Xi and an estimate for the probability distribution of the loss based on the empirical distribution. There are no assumptions on the probability distribution of Xi made, hence it is robust to non-normality and does not affect the results. For a confidence level α = 99%, we can compute the $VaR_{0.99}(L)$ and $ES_{0.99}(L)$ for the portfolio loss using historical estimation method in the following steps:

(1) At time t, collect the historical dataset $(X^i_{t-(n+1)\Delta}, \ldots, X^i_{t-2\Delta}, X^i_{t-\Delta}, X^i_t)$, for i =1,2,3,4 and where n = 2716 for this portfolio sample size and Δ = 1 day is the fixed distance between two consecutive time instants.

(2) Now use the historical estimation method to compute portfolio loss in terms of the Xi using the following formula:

$$L_{t+1} = -\Big(\sum_{i=1}^{4} \phi^i_t \cdot S^i_t (e^{X^i_{t+1}} - 1)\Big)$$

Which results in a dataset of losses: $\widetilde{L}_t = l_{[t]}\ (X^i_{t-(n+1)})$, where $l_{[t]}$ is the loss operator at time t. It is assumed that the losses during the different time periods are (i.i.d).

(3) Sort the absolute value of the losses in ascending order such that $\tilde{L}_{n,n} \leq \ldots \leq \tilde{L}_{1,n}$.

(4) Now, the $VaR_{0.99}(L)$ and $ES_{0.99}(L)$ can be computed using:

$$\widehat{VaR}_{0.99}(L) = q_L(0.99) = \tilde{L}_{[2716(1-0.99)]+1,2716} = \$84,785$$

$$\widehat{ES}_{0.99}(L) = \frac{\sum_{i=1}^{[2716(1-0.99)]+1} \tilde{L}_{i,n}}{[2716(1-0.99)]+1} = \$122,667.80$$

For VaR this means that our worst-case loss of our portfolio over 1 day is \$84,785 with a 99% confidence level. If VaR is exceeded then the maximum expected loss for the portfolio is \$122,667.80 with a 99% confidence level.

The other new method, which is called hybrid method in the paper by Boudoukh et al. (1998) [5], combines two approaches to VaR estimation: Risk Metrics[6] and historical estimation method, and hence hybrid method is partly alike historical estimation method. Just like the historical estimation method, the new method firstly orders the returns over a certain period from lowest to highest. Difference between two methods is that hybrid method assigns exponentially declining weights to historical returns while historical estimation method gives equal weights to each observed return. Meaning behind the new attributing method is that recent data have more impact on computing latest values of risk measures. Because of the different way of attributing weights, obtaining VaR at a certain significance level using same returns involves different number of observations for two methods. The exact number is

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up to how recently the extreme low returns are observed. What's more, historical estimation method does not consider linear interpolation usually while hybrid approach does.

Since the hybrid approach combines historical estimation method and exponential smoothing method, it inherits their advantages. Like historical estimation, hybrid method does not take assumptions before being used. Like exponential smoothing, the new approach uses exponential declining weights on past data, allowing users to capture the cyclical movement of return volatility and assigning more power to recent data, which is more realistic. It is worth mentioning that some empirical results presented that the new method provides an absolute error which is much lower than that of historical estimation approach and of exponential smoothing approach. Additionally, in the paper by Boudoukh et al. (1998), the author did a test and the results showed that the new method offers the lowest level of autocorrelation between assets and parameters among the three approaches. A very important advantage is that hybrid method provides improvements on the basis of historical estimation and exponential smoothing methods without increasing computational complexity, data intensity and programming difficulty.

The role of λ: In computation of time weights, λ is actually a decay factor. In the initial step of hybrid method, we denote l_t as the realized loss from t-1 to t. For each of the most recent K losses, l_t, l_{t-1},...,$l_{t-(K-1)}$we assign the weights:

$$\frac{1-\lambda}{1-\lambda^K}, \frac{1-\lambda}{1-\lambda^K}\cdot\lambda, ..., \frac{1-\lambda}{1-\lambda^K}\cdot\lambda^{K-1}$$

Where λ ∈ (0,1) is fixed. As we can see, when λ gets larger in interval (0,1), the weights given to data observed further in the past will be smaller while that assigned to more recent data will be greater. That means returns observed recently have more power to affect the risk measurement.

We choose λ = 0.98 when implementing the algorithm. The reason is that Boudoukh et al. (1998) fixed λ to be 0.98 when he gave an example for how to use hybrid method. Besides, figure 4 shows that decay factors for equity indices across different countries are close to 0.98 and because stocks are close to equity indices to some extent, we take equity indices as representative here. And When we take 99% as the value of the confidence level α, and let λ = 0.98, the results of $VaR_{0.99}(L)$ and $ES_{0.99}(L)$ are $57,863.65 and $66,129.70 respectively.

Optimal decay factors based on volatility forecasts
based on RMSE criterion

Country	Foreign exchange	5-year swaps	10-year zero prices	Equity indices	1-year money market rates
Austria	0.945	—	—	—	—
Australia	0.980	0.955	0.975	0.975	0.970
Belgium	0.945	0.935	0.935	0.965	0.850
Canada	0.960	0.965	0.960	—	0.990
Switzerland	0.955	0.835	—	0.970	0.980
Germany	0.955	0.940	0.960	0.980	0.970
Denmark	0.950	0.905	0.920	0.985	0.850
Spain	0.920	0.925	0.935	0.980	0.945
France	0.955	0.945	0.945	0.985	—
Finland	0.995	—	—	—	0.960
Great Britain	0.960	0.950	0.960	0.975	0.990
Hong Kong	0.980	—	—	—	—
Ireland	0.990	—	0.925	—	—
Italy	0.940	0.960	0.935	0.970	0.990
Japan	0.965	0.965	0.950	0.955	0.985
Netherlands	0.960	0.945	0.950	0.975	0.970
Norway	0.975	—	—	—	—
New Zealand	0.975	0.980	—	—	—
Portugal	0.940	—	—	—	0.895
Sweden	0.985	—	0.980	—	0.885
Singapore	0.950	0.935	—	—	—
United States	—	0.970	0.980	0.980	0.965
ECU	—	0.950	—	—	—

Figure 4. Optimal decay factors based on volatility forecasts (Risk Metrics technology document by JP Morgan, 1996)

3. The Capital Requirement of the Portfolio

We consider using formulas $C = \beta * VaR_{0.99} + (1-\beta)ES_{0.99}$ to calculate the capital requirement of the portfolio, then we identify whether the capital requirement C is a risk measure or not and shows its properties (For simplicity, let β=0.3)[7] [8].

3.1 Monotonicity

The capital requirement C is a Risk measure. Let L1 and L2 be two random variables, we can get that:

$$C(L^1) = 0.3VaR_{0.99}(L^1) + 0.7ES_{0.99}(L^1)$$

$$C(L^2) = 0.3VaR_{0.99}(L^2) + 0.7ES_{0.99}(L^2)$$

We know that VaR_α and ES_α are risk measure, which means they satisfied with monotonicity. Thus if L1 ≤ L2, it is easy to know:

$$0.3VaR_{0.99}(L1) + 0.7ES_{0.99}(L1) \leq 0.3VaR_{0.99}(L2) + 0.7ES_{0.99}(L2)$$

$$C(L^1) \leq C(L^2)$$

Thus the capital requirement C satisfies Monotonicity, and the capital requirement C is a Risk measure.

3.2 Translation Invariance

The capital requirement C is a Monetary Risk measure. Let L be a random variable, fix m ∈ R, we can get

$$C(L + m) = 0.3VaR_{0.99}(L + m) + 0.7ES_{0.99}(L + m)$$

As we know, VaRα and ESα are translation invariance, so:

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$C(L+m) = 0.3VaR_{0.99}(L+m) + 0.7ES_{0.99}(L+m)$

$= 0.3(VaR_{0.99}(L)+m) + 0.7(ES_{0.99}(L)+m)$

$= 0.3VaR_{0.99}(L) + 0.3m + 0.7ES_{0.99}(L) + 0.7m$

$= 0.3VaR_{0.99}(L) + 0.7ES_{0.99}(L) + m$

$= C(L) + m$

Thus the capital requirement C satisfies Translation invariance, and the capital requirement C is a Monetary Risk measure.

3.3 Convex risk Measure

The capital requirement C is not a Convex risk measure. Let L^1 and L^2 be two random variables, fix $\lambda \in R$, we can get:

$$C(\lambda L^1 + (1-\lambda)L^2) = 0.3VaR_{0.99}(\lambda L^1 + (1-\lambda)L^2) + 0.7ES_{0.99}(\lambda L^1 + (1-\lambda)L^2).$$

As we know, ESα satisfies Convexity but V aRα not. Thus we cannot get $C(\lambda L^1 + (1-\lambda)L^2) \le \lambda C(L^1) + (1-\lambda)C(L^2)$, the capital requirement C does not satisfy Convexity, and the capital requirement C is not a Convex risk measure.

3.4 Coherent risk Measure

The capital requirement C is not a Coherent risk measure, but it statisfies positively homogeneous. Let L be a random variable, fix $c \in (0,+\infty)$, we can get:

$$C(cL) = 0.3VaR_{0.99}(cL) + 0.7ES_{0.99}(cL)$$

As we know, V aRα and ESα are both Positive Homogeneity, so:

$C(cL) = 0.3VaR_{0.99}(cL) + 0.7ES_{0.99}(cL)$

$= 0.3cVaR_{0.99}(L) + 0.7cES_{0.99}(L)$

$= cC(L),$

thus the capital requirement C satisfies Positive Homogeneity. But as the capital requirement C is not a Coherent risk measure, as the capital requirement C does not satisfy Convexity[4].

4. Monte Carlo Methods

In this part, the estimations of $VaR_{0.99}(L_{t+1})$, $ES_0.99(L_{t+1})$ and $C(L_{t+1})$ by using Monte Carlo simulation method are presented. And we compare these estimations with the results of historical estimation method. The remainder of this thesis is about that we assume the dependence structure between risk factors of the vector X_{t+1} can be described as Student's t-copula instead of Gaussian copula. Then we apply a discount factor 0.9 to estimate the VaR and ES. At last, with a calibrated Student's t-copula and 10000 simulations, we compute a new capital C. We assume that the vector X_{t+1} of risk factor changes between t and t+1 follows a multivariate normal distribution:

$$X_{t+1} \sim N(\mu,\Sigma)$$

In this case, μ is the mean vector of X_{t+1}, Σ is the covariance matrix for X_{t+1}. We did 10,000 simulations, and find out $VaR_{0.99}(L_{t+1}) = 75454.38$, $ES_{0.99}(L_{t+1}) = 85630.77$ so that we got the result:

$$C = 0.9\big(0.3VaR_{0.99}(L_{t+1}) + 0.7ES_{0.99}(L_{t+1})\big) = \$77,725.65$$

We can find that the $VaR_{0.99}(L_{t+1})$ and $ES_{0.99}(L_{t+1})$ is bigger than multivariate normal distribution's results, cause the student-t copula has a sharper peak and heavier tail than the normal distribution, which means that as a result, the probability of extreme values is greater than that of normally distribution. Also, this Monte Carlo method can be used for various distributions of risk factors X_{t+1}, and it has strong ability to deal with nonlinear and non-normal problems. So that it can be applied flexible when model changes, and we can apply more models to fit the data to do a more reasonable prediction. [9]

5. Conclusion

In this paper, we presented three different methods for estimating VaR and ES of our small portfolio of four stocks during a given time period. The implementation process of each approach was discussed along with their advantages and disadvantages. This was done for the purpose was calculate the regulatory capital requirement for the investment portfolio, which the capital our investment portfolio needs to hold in reserves to have adequate protection against the risks we take on as an investment fun and shocks in the economy.

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